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The System of Two Simultaneous Ternary Quadratic Forms.

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Prof. Gordan has shown that all invariants and covariants of a single binary form or of simultaneous binary forms can be expressed as rational, integral functions of a certain set of invariants and covariants, finite in number, of the ground form or forms.* This set he calls the "*system*" of the ground forms. An essential advantage of the method that he employs is that it also furnishes, simultaneously with the proof of the theorem, the system itself; for he actually establishes a set of invariants (I include under "invariants" in this article covariants, contravariants, etc.), finite in number and having the property that every invariant can be expressed rationally and integrally in terms of these invariants. This set may contain superfluous members, namely, members that are themselves expressible rationally and integrally in terms of other members of the set. Such members are to be discarded, and the remaining members then constitute the system of the ground forms.

The method employed for the binary forms Gordan then extended to the ternary forms.† He discovered the characteristic marks of a set of invariants that form the system and showed how actually to find such a set. It may be remarked that there is a certain degree of choice as to the particular invariants that shall be taken to form the system. That this system would always be finite he did not prove generally, but he examined a number of special cases,‡ in all of which he found that the system is finite. Thus he established the system of the ternary cubic and biquadratic, and showed further, both for ternary and for quaternary forms, that if any number of ground forms each have finite systems, the system of these forms, considered as simultaneous, is also finite.

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* *Ueber das Formensystem binärer Formen*, Leipzig, 1875, in which the earlier demonstrations given in Crelle, 69, and Math. Ann., 3, are revised and simplified. See also Gordan-Kerschensteiner, *Invariantentheorie*, II, Leipzig, 1887, and Clebsch, *Binäre Formen*, Leipzig, 1872.

† *Ueber ternäre Formen dritten Grades*, Math. Ann., Bd. 1.

‡ *Ueber das Formensystem*, etc., p. 50. See also Math. Ann., Bd. 17, p. 217.

Recently, Dr. Hilbert[*] has proved the general theorem that the system of an n-ary form or of such simultaneous forms is finite. How to find this system is, however, a question that at present has not been answered.

The object of the present paper is to show, in an elementary manner, how Gordan established the system of two simultaneous ternary quadratic forms. An account of this method is given in Clebsch-Lindemann, *Geometrie*, I, Leipzig, 1876, p. 288, but for the reader that is unacquainted with Gordan's methods this account is too brief to be easily intelligible. The importance of these investigations, which at once form a natural introduction to the theory of the ternary forms and help to systematize the geometric theory of a system of two conics, renders it desirable that they should be easily accessible. The notation employed is the symbolic notation of Aronhold and Clebsch, the essential properties of which I have explained in the present number of this Journal (*The Symbolic Notation of Aronhold and Clebsch*).

§1.—*The Symbolic Representation of Invariants of f and f'.*

Let the two simultaneous ternary quadratic forms be

$$\left.\begin{array}{l} f = a_x^2 = b_x^2 = c_x^2 = \ldots. \\ f' = a_x'^2 = b_x'^2 = c_x'^2 = \ldots. \end{array}\right\} \quad (1)$$

and

Then surely the discriminants[†]

$$A_{111} = (abc)^2, \quad A_{222} = (a'b'c')^2$$

of f and f', as also the forms in u which, put equal to 0, are the equations in line coordinates of the conics $f = 0$, $f' = 0$:

$$F_{11} = (abu)^2, \quad F_{22} = (a'b'u)^2$$

must belong to the system.[‡]

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[*] *Ueber die Theorie der algebraischen Formen*, Math. Ann., Bd. 36.

[†] Cf. *Symb. Not.* §3.

[‡] See Clebsch-Lindemann, *Geometrie*, I, p. 287, where it is shown that the system of a single ternary quadratic form $f = a_x^2 = b_x^2 = \ldots.$ consists of the form itself, the discriminant $(abc)^2$ and the form $(abu)^2$, the vanishing of which gives the equation of the conic $f = 0$ in line coordinates. The identical invariant u_x belongs to every system and, therefore, is not further explicitly mentioned.

Whenever two symbols of f, as a and b, occur in the combination

$$(a_2b_3 - a_3b_2) = (ab)_1, \quad (a_3b_1 - a_1b_3) = (ab)_2, \quad (a_1b_2 - a_2b_1) = (ab)_3,$$

it is of practical advantage to replace them by a single symbol α:

$$\alpha_1 = (ab)_1, \quad \alpha_2 = (ab)_2, \quad \alpha_3 = (ab)_3,$$

or, abbreviated, $\alpha = (ab)$. Thus

$$(abu) = u_1\alpha_1 + u_2\alpha_2 + u_3\alpha_3 = u_\alpha.$$

The symbols a, b are cogredient with the line coordinates u when the linear transformation

$$x_i = \xi_i x_1' + \eta_i x_2' + \zeta_i x_3', \quad i = 1,\ 2,\ 3$$

is performed,* but the combinations (ab) of a and b, *i. e.* the symbols α, are cogredient with the point coordinates x, and hence the notation u_α, which suggests an analogy between α and x, is justified. We now have

$$A_{111} = a_\alpha^2, \quad A_{222} = a_{\alpha'}'^2, \quad F_{11} = u_\alpha^2, \quad F_{22} = u_{\alpha'}^2, \tag{2}$$

where, in the first expression, $\alpha = (bc)$.

The simplification effected by the introduction of the symbols α, β, ... appears from the following theorem: *if the symbols a, b appear once in the symbolic product Π in the combination (ab):*

$$\Pi = (abv)M = v_\alpha M,$$

where v is any element cogredient with u, then their further appearance will also be in the combination (ab), so that

$$\Pi = v_\alpha w_\alpha M',$$

where M' no longer contains a, b, and hence they can be completely replaced by the single symbol α. For either Π is already of the form demanded by the theorem, or else:

$$\Pi = (abv)a_s b_t M_1,$$

where a, b now enter only explicitly. (s, t are, for example, (cd), $(c'd)$, $(c'd')$, (cu), $(c'u)$, etc., or x; in general, they are any combinations of symbols and coordinates that are cogredient with x. In the same way, v, w are symbols c, d, a', b', or u, or any combinations of symbols and coordinates that are cogredient with u.) When a, b are interchanged, the value of Π is unaltered, the form of Π becomes:

$$\Pi = (bav)b_s a_t M_1 = -(abv)a_t b_s M_1.$$

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* Cf. *Symb. Not.* §3.

Adding these two expressions for Π together,

$$2\Pi = (abv)(a_s b_t - a_t b_s)M_1 .$$

But, by identity VII,*

$$a_s b_t - a_t b_s = (abw), \quad w = (st),$$

—the elements w are cogredient with u and hence the notation is justified—and it appears that

$$\Pi = \tfrac{1}{2}(abv)(abw)M_1 = v_\alpha w_\alpha \cdot \tfrac{1}{2}M_1 . \qquad \text{Q. E. D.}$$

We will, therefore, in the following always, when possible, replace symbols a, b, ... by symbols α, β, ... The above applies equally well to the replacement of symbols a', b', ... by symbols α', β', ...

A corresponding theorem holds for products in which the symbols α, β appear in the combination $(\alpha\beta)_i$, $i = 1, 2, 3$. For, if the symbolic product Π contains the factor $(\alpha\beta s)$, then Π either contains a second factor $(\alpha\beta t)$ or else is of the form:

$$\Pi = (\alpha\beta s)M = (\alpha\beta s)v_\alpha w_\beta M',$$

where M' is free from α, β; and then

$$2\Pi = (\alpha\beta s)(v_\alpha w_\beta - v_\beta w_\alpha)M_1 .$$

By identity VIII, $\quad v_\alpha w_\beta - v_\beta w_\alpha = (\alpha\beta t), \quad t = (vw),$

and Π is also in this case of the form:

$$\Pi = (\alpha\beta s)(\alpha\beta t) \cdot \tfrac{1}{2}M_1 .$$

All rational integral invariants of f and f' are formed of terms that are products of symbolic factors of the types†

$$(abc), \quad (aba'), \quad (aa'b'), \quad (a'b'c'), \quad (abu), \quad (aa'u), \quad (a'b'u), \quad a_x, \quad a'_x .$$

Introducing the symbols α, α', we must annex to the foregoing further the symbolic factors $(\alpha\beta x)$, $(\alpha'\beta' x)$, $(\alpha\alpha' x)$, and hence we see that *every rational integral invariant of f and f' is a sum of products of factors of the following types:*

$$\left.\begin{array}{llll} a_x, & a'_x, & u_\alpha, & u_{\alpha'}, \\ a_\alpha, & a'_\alpha, & a_{\alpha'}, & a'_{\alpha'}, \\ (aa'u), & (\alpha\beta x) & (\alpha'\beta' x), & (\alpha\alpha' x), \end{array}\right\} \qquad (3)$$

each product multiplied by a numerical constant.

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* The identities referred to in this paper are those given in *Symb. Not.*, §6.

† Cf. *Symb. Not.*, §4.

§2.—*The Method of Solution of the Problem of the Present Paper.*

Before attempting to describe the general method of solution of the problem of this paper, I will illustrate the method by a few simple examples.

a). A symbolic product Π of factors of the types (3) that contains the symbolic factor a_α contains the actual factor $a_\alpha^2 = A_{111}$. For Π is either of the form

$$\Pi = (abc)a_r b_s c_t \,.\, M \quad \text{or} \quad = (abc)(abv)c_t \,.\, M,$$

where M does not contain the symbols a, b, c, and r, s, t and v have such values as in the previous paragraph. By identity VI,

$$a_r b_s c_t + a_t b_r c_s + a_s b_t c_r - a_r b_t c_s - a_s b_r c_t - a_t b_s c_r = (abc)(rst).$$

Multiplying each side of this identity by $(abc) \,.\, M$, we see that each of the six expressions on the left hand side represents Π, and hence

$$6\Pi = (abc)^2(rst)M, \quad \Pi = A_{111} \,.\, \tfrac{1}{6}(rst)M.$$

In the second case, by identity IV,

$$(abv)c_t - (cbv)a_t - (acv)b_t = (abc)v_t.$$

Multiplying by $(abc)M$, we have

$$3\Pi = (abc)^2 v_t M, \quad \Pi = A_{111} \,.\, \tfrac{1}{3}v_t M.$$

Similarly for products containing the symbolic factor $a'_{\alpha'}$: such products contain the actual factor A_{222}.

b). Again, if a symbolic product Π contains a factor of the type $(\alpha\beta s)$ and hence (§1) also the symbolic factor $(\alpha\beta t)$:

$$\Pi = (\alpha\beta s)(\alpha\beta t)M,$$

then Π contains the factor A_{111}. (If the constituents of Π were all factors of the types (3), s could be only x; but it is often convenient, as will appear later, to consider products in which $s = (au)$, $(a'u)$, (aa'), etc.) For,

$$(\alpha\beta s) = \begin{vmatrix} (ab)_1 & (ab)_2 & (ab)_3 \\ (cd)_1 & (cd)_2 & (cd)_3 \\ s_1 & s_2 & s_3 \end{vmatrix} = (abd)c_s - (abc)d_s,$$

$$(\alpha\beta s)^2 = 2(abc)^2 d_s^2 - 2(abc)(abd)c_s d_s,$$

since $(abd)^2 c_s^2 = (abc)^2 d_s^2$. But the second term can be reduced to simpler form, for, by interchanging equivalent symbols, we find that

$$(abc)d_s\,.\,(abd)c_s = \tfrac{1}{3}(abc)d_s\{(abd)c_s - (cbd)a_s - (acd)b_s\} = \tfrac{1}{3}(abc)^2d_s^2$$

by identity IV, and hence

$$(\alpha\beta s)^2 = \tfrac{4}{3}(abc)^2d_s^2 = \tfrac{4}{3}A_{111}a_s^2.$$

The polar operator, $\tfrac{1}{2}\left(t_1\frac{\partial}{\partial s_1} + t_2\frac{\partial}{\partial s_2} + t_3\frac{\partial}{\partial s_3}\right)$, applied to each side of this equation, gives

$$(\alpha\beta s)(\alpha\beta t) = \tfrac{4}{3}A_{111}a_sa_t.$$

And similarly,

$$(\alpha'\beta' s)(\alpha'\beta' t) = \tfrac{4}{3}A_{222}a'_sa'_t.$$

Thus it appears that every symbolic product, Π, that contains any one of the symbolic factors a_α, $a'_{\alpha'}$, $(\alpha\beta s)$, $(\alpha'\beta' s)$, breaks up into two actual factors, one of which (A_{111}, A_{222}) belongs to those invariants (2) that we have already admitted to the system, and the other of which (M) contains fewer symbols than Π; and hence a symbolic product will surely be reducible to simpler products if it contains other factors than those of the following types:

$$\left.\begin{array}{llll} a_x, & a'_x, & u_\alpha, & u_{\alpha'}, \\ & a_{\alpha'}, & a'_\alpha, & \\ & (aa'u), & (\alpha\alpha' x). & \end{array}\right\} \qquad (4)$$

The method of establishing the system of f and f' consists now in ascertaining still further restrictions on the choice of symbols for a symbolic product that shall really be an irreducible invariant, and not a product of two invariants, one (or both) of which contains fewer symbols than Π, or a sum of such products. It will appear that the number of such irreducible products is very limited, and these, taken with (1') and (2), form a set of invariants, in terms of which every symbolic product, and hence every simultaneous invariant of f and f', can be expressed rationally and integrally, *i. e.* they form the system of f and f'.

§3.—*Further Reduction of Symbolic Products.*

I will now show that a symbolic product containing two or more cogredient symbols of f: a, b, ... *or* α, β, ..., *i. e.* a product Π of the form

$$\Pi = a_ra_sb_tb_zM \quad \text{or} \quad = \rho_\alpha\sigma_\alpha\tau_\beta\omega_\beta M,$$

is expressible in terms of symbolic products each containing fewer symbols than Π. The same proof applies to products containing two or more cogredient symbols of f': a', b', ... *or* α', β', ... Hence it follows that all symbolic products can be expressed in terms of such as contain at most one of each of the symbols a, α, a', α'.

Theorem A.—In the symbolic products,

$$\Pi = a_r a_s b_t b_z M, \tag{I}$$

$$\Pi = \rho_\alpha \sigma_\alpha \tau_\beta \omega_\beta M, \tag{II}$$

either two of the elements r, s, t, z *or* ρ, σ, τ, ω *are identical or else* Π *can be expressed in terms of products for which this is the case, and of products that contain fewer symbols than* Π. *And similarly for products containing two or more symbols of* f'.

First consider the product (I). From the last § it is evident that we may restrict our attention to those cases in which r, s, t, z are four of the elements (see scheme (4)):

$$r,\ s,\ t,\ z: \begin{cases} x, \\ \alpha',\ \beta',\ \gamma',\ \ldots . \\ (a'u),\ (b'u),\ \ldots . \end{cases} \tag{5}$$

In no case can more than one of the elements r, s, t, z belong to the last row of (5); for otherwise Π would contain one of the two factors:

$$a_r a_s b_t b_z = \begin{cases} (aa'u)(ab'u)b_t b_z \\ (aa'u)(bb'u)a_t b_z, \end{cases}$$

where t, z are any two of the elements of (4). In the first case Π would be of the form:

$$\Pi = a'_{r'} a'_{s'} b'_{t'} b'_{z'} M, \quad \text{where} \quad r' = t' = (au),$$

and would, therefore, satisfy the conditions of the theorem, enunciated for the symbols of f'. In the second case, the application of identity IV:

$$(bb'u)a_t = (b'ua)b_t - (uab)b'_t + (abb')u_t$$

gives

$$(aa'u)(bb'u)a_t b_z = (aa'u)(ab'u)b_t b_z - (aa'u)(abu)b_z b'_t + (aa'u)(abb')b_z u_t.$$

The first of these symbolic products is of the form just considered; in the second and third, a, b can be replaced by the single symbol α and the symbolic products arising from these terms will contain fewer symbols than Π. Hence the theorem is proved for all products, Π, in which two or more of the symbols r, s, t, z belong to the last row of (5).

After the exclusion of such products, there remain to be considered products of the following three types:

$$a_r a_s b_t b_z = \begin{cases} a_{\alpha'} a_{\beta'} b_{\gamma'} b_z \\ a_{\alpha'} a_{\beta'} b_x b_z \\ a_{\alpha'} a_z \, b_x b_{\beta'} \end{cases}$$

where z is arbitrary. The first and second of these products already satisfy the conditions of the theorem, for Π is then of the form:

$$\Pi = \rho_{\alpha'} \sigma_{\alpha'} \tau_{\beta'} \omega_{\beta'} M, \quad \text{where} \quad \rho = \tau = a.$$

The third is also of a form already considered, unless $z = (a'u)$. Then

$$\Pi = (aa'u) a_{\alpha'} b_\beta \, b_x \cdot \rho_{\alpha'} \sigma_{\beta'} M,$$

where ρ, σ are of the following types: u, c, d, $[c', d',]$ (αx), (βx), $[(\gamma' x), (\delta' x)]$. For the bracketed symbols Π would be directly reducible (§§1 and 2). If $\rho = (\alpha x)$, then Π contains the factor (identity VI):

$$(aa'u)(\alpha\alpha' x) = \begin{vmatrix} a_\alpha & a'_\alpha & u_\alpha \\ a_{\alpha'} & a'_{\alpha'} & u_{\alpha'} \\ a_x & a'_x & u_x \end{vmatrix} \tag{6}$$

and is, by the foregoing, reducible. Hence ρ, σ can be only of the form u, c, d. One of them, ρ, let us say, must, therefore, $= c$, and Π contains the factor $a_s a_{\alpha'} c_{\alpha'} c_t$, where $s = (a'u)$ and t may have any value. This product satisfies the conditions of the theorem, and thus the proof of the theorem for products of the form (I) is complete.

It remains to consider products of the form (II). This is the dualistic of the case just considered and the treatment is accordingly analogous. The reductions differ, however, in certain details from the previous ones, and for this reason I give them at length.

Only products in which ρ, σ, τ, ω are four of the elements:

$$\rho, \sigma, \tau, \omega: \begin{cases} u, \\ a', b', c' \dots. \\ (\alpha' x), (\beta' x), \dots. \end{cases} \tag{7}$$

need be considered. Not more than one element, ρ, can belong to the last row, for otherwise we should have

$$\rho_\alpha \sigma_\alpha \tau_\beta \omega_\beta = \begin{cases} (\alpha\alpha' x)(\alpha\beta' x) \tau_\beta \omega_\beta \\ (\alpha\alpha' x)(\beta\beta' x) \tau_\alpha \omega_\beta \end{cases}.$$

In the first case, Π has the factor $\rho'_{\alpha'}\rho'_{\beta'}$ and thus satisfies the conditions of the theorem; in the second case (identity IV)*

$$(\alpha\alpha' x)\omega_\beta \,.\, (\beta\beta' x)\tau_\alpha = (\alpha\alpha' x)\omega_\beta\{(\beta' x\alpha)\tau_\beta - (x\alpha\beta)\tau_{\beta'} + (\alpha\beta\beta')\tau_x\}.$$

The first term is of the form just considered; the second and third have the factor $(\alpha\beta x)$ or $(\alpha\beta\beta')$, and these latter terms contain accordingly (§§1, 2) the actual factor A_{111}; they are, therefore, expressible in terms of products that contain fewer symbols than Π.

Lastly, when at least three of the elements ρ, σ, τ, ω belong to the first two rows of (7),

$$\rho_\alpha\sigma_\alpha\tau_\beta\omega_\beta = \begin{cases} a'_\alpha b'_\alpha c'_\beta\omega_\beta \\ a'_\alpha b'_\alpha u_\beta\omega_\beta \\ a'_\alpha\omega_\alpha u_\beta b'_\beta \end{cases}$$

where ω is arbitrary. The first and second of these products are already of the form demanded by the theorem, the third needs to be considered only when $\omega = (\alpha' x)$. Π then contains the factor

$$(\alpha\alpha' x)a'_\alpha b'_\beta u_\beta \,.\, a'_r b'_s,$$

where r, s are of the following types: x, γ, δ, $[\gamma', \delta',]$ (au), (bu), $[(c'u), (d'u)]$, and it is evident from similar considerations to those of the previous case that the theorem holds for this product.

The proof of the theorem for products in the symbols of f is thus complete. Interchanging primed and unprimed symbols, we obtain at once the proof for the corresponding products in the symbols of f'.

Theorem B.—Symbolic products

$$\Pi = a_r a_s b_t b_z M, \tag{I}$$
$$\Pi = \rho_\alpha\sigma_\alpha\tau_\beta\omega_\beta M, \tag{II}$$

in which two of the symbols r, s, t, z *or* ρ, σ, τ, ω *are identical, are expressible in terms of symbolic products containing fewer symbols than* Π. *The same also holds for similar products in the symbols of* f'.

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* It is to be noticed that symbolic determinant factors have the invariant property only when all the symbols are cogredient. Hence when a, b, c of identity IV are cogredient with x, the identity must be written in the form

$$(rst)d_x - (stx)d_r + (txr)d_s - (xrs)d_t \equiv 0.$$

For products of the form (I) there are three cases to consider:

$$\begin{array}{ll} 1) \quad r = t = x & \begin{cases} \text{a)} \quad s = \alpha' \\ \text{b)} \quad s = (a'u) \end{cases} \\ 2) \quad r = t = \alpha' & \\ 3) \quad r = t = (a'u) & \end{array}$$

$$\begin{array}{ll} 1) \text{ a)} & a_x a_{\alpha'} b_x b_z = a_x b_z\left(a_x b_{\alpha'} + (\alpha\alpha' x)\right) \\ \quad\ \text{b)} & (aa'u) a_x b_x b_z = a_x b_z \{ (a'ub) a_x - (uba) a'_x + (baa') u_x \} \\ 2) & a_{\alpha'} a_s b_{\alpha'} b_z = a_{\alpha'} b_z \{ a_{\alpha'} b_s + (\alpha s \alpha') \} \\ 3) & (aa'u)(ba'u) a_s b_z = (ba'u) a_s \{ (a'ub) a_z - (uba) a'_x + (baa') u_x \} \end{array}$$

by the help of identities IV and VII. The terms on the right hand side either contain as factors terms with a less number of symbols than Π or admit the replacement of the symbols a, b by the single symbol α, and hence meet in every case the demands of the theorem.

It is not necessary to make these divisions into cases, for the reductions for all four cases are comprised in the single formula:

$$a_r a_s b_r b_z = a_r b_z \{ a_r b_s + (\alpha s r) \} . \tag{8}$$

Products of the form (II) correspond dualistically to those just considered and are treated in a similar manner. As before, three cases present themselves:

$$\begin{array}{ll} 1) \quad \rho = \tau = u & \begin{cases} \text{a)} \quad \sigma = \alpha' \\ \text{b)} \quad \sigma = (\alpha' x) \end{cases} \\ 2) \quad \rho = \tau = \alpha' & \\ 3) \quad \rho = \tau = (\alpha' x) . & \end{array}$$

The desired reduction in each case is given by the formula:

$$\rho_\alpha \sigma_\alpha \rho_\beta \tau_\beta = \rho_\alpha \tau_\beta \{ \sigma_\beta \rho_\alpha + (\alpha \beta t) \} , \quad \text{where} \quad t = (\sigma\rho) , \tag{9}$$

and the theorem is proved.

By the repeated application of these two theorems, it is evident that every symbolic product can be ultimately expressed in terms of symbolic products, no one of which contains two cogredient symbols of f or two cogredient symbols of f'.

§4.—*The System of f and f'.*

It is now an easy task to write down a set of invariants in terms of which every other rational integral simultaneous invariant of f and f' is expressible rationally and integrally. A set of symbolic products, each of which contains at most *one* symbol of each of the four types a, b . . . . , α, β . . . . , a', b' . . . . , α', β' . . . . is such a set, and is obtained, with the exception of two of its members, $A_{111} = a_\alpha^2$ and $A_{222} = a'^2_{\alpha'}$, by combining the symbolic factors (4) with each other in all possible ways. But the set thus obtained may be restricted still further, for from identity VI (see (6)) it is clear that any such product having the factor $(aa'u)(\alpha\alpha'x)$ can be replaced by products that do not contain these two determinant factors, and hence such products may be excluded from the set. The symbolic products that now remain are the following:

$$\left.\begin{array}{rlrl}
f = a_x^2 & , & f' = a'^2_x & , \\
F_{11} = u_\alpha^2 & , & F_{22} = u_{\alpha'}^2 & , \\
A_{111} = a_\alpha^2 & , & A_{222} = a'^2_{\alpha'} & , \\
A_{112} = a'^2_\alpha & , & A_{122} = a_{\alpha'}^2 & , \\
B_1 = a'_\alpha a'_x u_\alpha & , & B_2 = a_{\alpha'} a_x u_{\alpha'} & , \\
 & & & \\
N = (aa'u) a_x a'_x & , & \mathrm{N} = (\alpha\alpha'x) u_\alpha u_{\alpha'} & , \\
C_1 = (aa'u) a'_\alpha a_x u_\alpha & , & \Gamma_1 = (\alpha\alpha'x) a_{\alpha'} u_\alpha a_x & , \\
C_2 = (aa'u) a_{\alpha'} a'_x u_{\alpha'} & , & \Gamma_2 = (\alpha\alpha'x) a'_\alpha u_{\alpha'} a'_x & , \\
D = (aa'u) a_{\alpha'} a'_\alpha u_\alpha u_{\alpha'} & , & \Delta = (\alpha\alpha'x) a_{\alpha'} a'_\alpha a'_x a_x & , \\
 & & & \\
F_{12} = (aa'u)^2 & , & \Phi_{12} = (\alpha\alpha'x)^2 & .
\end{array}\right\}$$

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f and f'.

No one of these twenty invariants is expressible rationally and integrally in terms of the others and of the identical invariant u_x. *These invariants form the system of f and f'.*

Between these invariants exist certain identical relations of higher order.[*] Thus it is easy to show[†] that

$$u_x D \equiv F_{11} C_2 + F_{22} C_1 - \mathrm{N} F_{12}$$

and, dualistically,

$$u_x \Delta \equiv f \Gamma_2 + f' \Gamma_1 - N \Phi_{12}.$$

For, the first of these identities is, symbolically written,

$$(aa'u) a_{\alpha'} a'_\alpha u_\alpha u_{\alpha'} u_x \equiv (aa'u) a_{\alpha'} a'_x u_\alpha^2 u_{\alpha'} + (aa'u) a'_\alpha a_x u_{\alpha'}^2 u_\alpha - (aa'u)^2 (\alpha\alpha'x) u_\alpha u_{\alpha'},$$

or

$$(aa'u) u_\alpha u_{\alpha'} \{ (aa'u)(\alpha\alpha'x) + a_{\alpha'} a'_\alpha u_x - a_{\alpha'} a'_x u_\alpha - a'_\alpha a_x u_{\alpha'} \} \equiv 0 .$$

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[*] Gordan: *Über Büschel von Kegelschnitten,* Math. Ann., Bd. 19.

[†] Forsyth: *Systems of Ternariants that are Algebraically Complete,* Amer. Jour., Vol. 12, p. 53.

Now, by identity VI,

$$(aa'u)(\alpha\alpha'x) - a_\alpha a'_{\alpha'} u_x - a_x a'_\alpha u_{\alpha'} - a_{\alpha'} a'_x u_\alpha + a_\alpha a'_x u_{\alpha'} + a_{\alpha'} a'_\alpha u_x + a_x a'_{\alpha'} u_\alpha \equiv 0,$$

and it is sufficient to show that

$$(aa'u) u_\alpha u_{\alpha'} \{ a_\alpha a'_{\alpha'} u_x - a_\alpha a'_x u_{\alpha'} - a_x a'_{\alpha'} u_\alpha \} \equiv 0.$$

This appears as follows. The first term has the symbolic factors a_α and $a'_{\alpha'}$, and is, therefore, (§2) of the form $A_{111} A_{222} u_x \,.\, M$. But M must vanish identically, for, as an invariant, it is a product of factors of the types $(\rho\sigma\tau)$, $(\rho\sigma u)$, u_x given in *Symb. Not.* §4, and out of the three u's that remain no such factors can be formed that do not vanish identically. And similarly for the other two terms; they contain respectively the factors A_{111} and A_{222}, and after these factors have been extracted, there are not enough elements left from which to form non-vanishing factors of the degree required. The above is, therefore, a true identity.

An extended discussion of the geometric signification of the invariants of the system for the conics f and f' is to be found in Clebsch-Lindemann, *Geometrie*, I, pp. 291–304; further simultaneous invariants of f and f' are there treated and expressed in terms of the invariants of the system.

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